LEARNING THE CONCLUSION MAKING METHODS IN THE INSTRUCTION IN THE NATURAL-MATHEMATICAL GROUP OF SUBJECTS

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Abstract. Training students to learn and solve problem situations on their own is one of the basic goals of the educational process. The successful realization of this complex task, among other things, implies that the students continuously learn about the conclusion making methods. This knowledge is essential for learning about the natural and social phenomena. The teachers must pay special attention to the acquisition of the conclusion making methods in the instruction. When carrying out the lessons, it is necessary to continuously learn about the conclusion making methods in the process of realization of specific instructional content. The content from mathematics and science can serve this goal. Bearing in mind the previously said, it is especially important when making the syllabi, one of the objectives to be the learning of the conclusion making methods. In this paper we are going to present short overview of the conclusion making methods, as well as examples how the students can learn them from the youngest age.

Keywords: conclusion making methods, instruction, learning, mathematics, subjects related to science.

1. INTRODUCTION

Thinking is an active process of cognition during which the real world reflects in the consciousness of people. For this reason, one of the most important objectives of the modern instruction is to develop the properties of thinking of the students. Thoughts can have a different structure. The development of individual thoughts and their individual combinations are called forms of thinking. The most important forms of thinking are: notions, claims and reasoning (concluding). Further on, we are going to analyze making conclusions as a form of thinking and discuss its role in the instructional process, paying special attention to how different types of making conclusions can be acquired in natural-mathematic group of subjects.

An essential instrument in the cognition process is the so-called logical concluding. It is used for carrying out researches, proving claims, systematization of knowledge, checking hypotheses, etc.

The claims that are used for creating new knowledge are called *assumptions (premises)*, and the new claim which is acquired by comparison or by combining assumptions is called *a conclusion (output).* Logical reasoning, i.e. making conclusions is an intellectual operation that leads to new knowledge. Its value for the process of cognition lies in the fact that it is used for acquiring new knowledge without experimentation or practical testing, thus expanding the cognition potentials. Further on, the conclusion from the reasoning can be *correct (true)* if it meets these two conditions:

1) The assumptions are true.
2) The laws of thinking are used in the right way with the assumptions in the process of the logical operation, i.e. in their comparison and connecting.

The reasoning in which the 2) condition is met is called *correct reasoning.* Failure to meet one of these conditions can lead to a false conclusion. The irregularity in the reasoning can be:

a) *Textual,* wrong use of words, mixing meanings of the same word (homonym).
b) *Logical,* a mistake in the properties of the thought or the form of relation between the claims in the reasoning (redundancy, continuous repetition of operations when proving, breaking the laws of logic).

Here we are not going to focus on the errors that may happen in the concluding but rather we are going to analyze analogy, inductive and deductive reasoning, as well as the ways in which they are acquired in science.

2. THE CONCLUSION MAKING METHODS IN THE INSTRUCTION IN THE NATURAL-MATHEMATICAL GROUP OF SUBJECTS

Induction, as *a method of reasoning* is a complex intellectual operation in which we start with some separate facts, and from two or more separate facts we form a general claim. This is the usual pattern for concluding by induction: $M = \{a_i \mid i \in I\}$ is a set, and P is a property of the elements of M. We write $P(x)$ if the element x has the property P, and $P(\bar{x})$ if the element x does not have the property P. The property P has been confirmed for the elements $a_i \in M, i = 1, 2, ..., k$. The inductive conclusion is carried out according to the following pattern.

$$\frac{P(a_1), P(a_2), ..., P(a_k)}{\text{Conclusion: probably } P(a) \text{ for each } a \in M}. \quad (1)$$

If the set M is finite and has k elements, than the formula

$$(\text{for each } x \in M,\ P(x)) \Leftrightarrow P(a_1) \wedge P(a_2) \wedge ... \wedge P(a_k) \quad (2)$$

is always true, and the pattern

$$\frac{P(a_1), P(a_2), ..., P(a_k)}{\text{Conclusion: } P(a) \text{ for each } a \in M}$$

is correct concluding, which is called *complete induction*.

If $|M| > k$, and M is an infinite set, i.e. the analyzed k cases do not cover all possible cases, and thus the conclusion from the pattern (1) does not necessarily have to be correct. It is just probable that it is correct. In this case, the concluding according to the pattern (1) is called *incomplete induction* or just *induction.*

The complete induction can also be used in certain cases when the set M is indefinite, if M can be separated into a finite number of disjunctive subsets, and if for each of them we can confirm the validity of a specific property of the elements of the set M.

Example (complete induction). If n is an even natural number from the second or third ten, then it is a sum of two prime numbers.

Let us look at the following cases:

$$12 = 5 + 7, \quad 14 = 7 + 7, \quad 16 = 3 + 13, \quad 18 = 5 + 13, \quad 20 = 7 + 13,$$
$$22 = 3 + 19, \quad 24 = 7 + 17, \quad 26 = 3 + 23, \quad 28 = 5 + 23, \quad 30 = 11 + 19$$

The set $M = \{12, 14, 16, ..., 30\}$ is finite and all cases have been tested. A conclusion was made that the claim is correct for all separate cases.

Example (incomplete induction). Let us calculate the remainder of the division of the number $4^n + 15n$ with 9, in which $n \in \mathbf{N}$.

For $n = 1$, we get $4^1 + 15 = 19 = 3 \cdot 6 + 1$, which means that the remainder is 1. For $n = 2$, we get $4^2 + 30 = 46 = 3 \cdot 15 + 1$, which means that the remainder is 1. For $n = 3$, we get $4^3 + 45 = 109 = 3 \cdot 36 + 1$, which means that the remainder is 1. It is natural to think when $n \in \mathbf{N}$, and $4^n + 15n$ is divided by 9, we get a remainder 1. In this case we use the pattern of making conclusions (1) and we get

$$\frac{P(1), P(2), P(3)}{\text{Conclusion: probably } P(n) \text{ for each } n \in \mathbf{N}}$$

We should note that each following case, for which the claim is true, supports our conclusion, but this does not mean that we have a proof for the devised conclusion.

Example (incomplete induction). We test the claim that all vertebrate have a similar structure of the skeleton of the front extremities. For this reason, we compare the structure of (fig. 1):

1. a frog
2. a crocodile
3. a whale
4. a cat
5. a bird
6. a bat
7. a human.

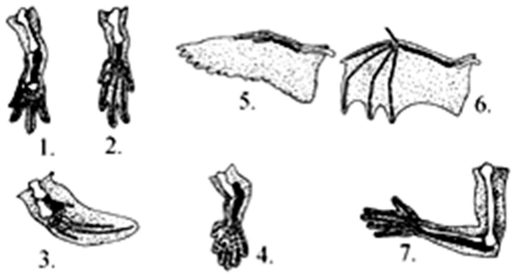

Fig. 1

We compared the skeletons of the front extremities of only seven different vertebrates and noticed that they have a similar structure. However, there are many other vertebrate species, and thus we say that this is incomplete induction, and our conclusion is stated in the following way:

It is probable that all vertebrates have similar structure of the front extremities.

The following example demonstrates that the incomplete induction does not have to lead towards a correct conclusion.

Example (incomplete induction). It is correct that for each $n = 1, 2, 3, \ldots, 15$, $n^2 + n + 17$ results in a prime number, but it is incorrect that for each $n \in \mathbf{N}$, the number $n^2 + n + 17$ is a prime number. Indeed, for $n = 16$, we get $n^2 + n + 17 = 17^2$, which is a composite number.

Deduction, as *a reasoning method,* is an intellectual activity with which from one or several correct claims, used as assumptions, we devise a new claim, which according to the rules of logic must be the result of the assumptions. Deductive concluding can be made from the general to the specific or individual, as well as from the individual to the specific. It is based on the laws of logic and the rules for output. It is interesting to note that the form, structure and validity of the content of the claim are important, which is not the case with the content. There are different laws and rules for output, which means that there are different patterns for deductive making concluding. Some of these patterns of concluding are acquired in the mathematics and the natural group of subjects in a subtle manner, because in all forms of instruction, even in the subtle ones, the following equivalents (laws of logic) are used:

1) $p \wedge q \Leftrightarrow q \wedge p$; $p \vee q \Leftrightarrow q \vee p$, (commutative law);

2) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$; $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$, (associative laws);

3) $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$; $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$, (distributive laws);

4) $\neg\neg p \Leftrightarrow p$, (law of double negation)

5) $p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$, (law of contrapositive);

6) $p \Rightarrow q \Leftrightarrow \neg p \vee q$;

7) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$; $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$;

8) $p \Rightarrow q \Leftrightarrow \neg q \wedge p \Rightarrow \neg p$;

9) $p \Rightarrow q \Leftrightarrow \neg q \wedge p \Rightarrow q$;

10) $p \Rightarrow q \Leftrightarrow \neg q \wedge p \Rightarrow r \wedge \neg r$

We are only going to add that the use of the previously mentioned and other laws is the best way for development of the deductive concluding. It is necessary that the students use the laws of logic in a cognitive way, which by all means can be done by simple examples, such as the following:

Example: Learning the rule for implication elimination (*modus ponens*), i.e. the statement formula

$$(p \Rightarrow q) \wedge p \Rightarrow q, \quad (3)$$

in which the statements $p \Rightarrow q$ and p are assumptions, and the statement q is a conclusion, in the primary education can be carried out with examples such as the following:

Assumptions. 1) When it rains, the street is wet.

2) It is raining.

Conclusion. The street is wet.

Example. Learning the rule *modus tolens,* i.e. the statement formula

$$(p \Rightarrow q) \wedge \neg q \Rightarrow \neg p \quad (4)$$

where the statements $p \Rightarrow q$ and $\neg q$ are *assumptions*, and the statement $\neg p$ is a conclusion, in primary education can be carried out with examples such as the following:

Assumptions. 1) When it rains, the street is wet.

2) The street is not wet.

Conclusion It is not raining.

Example. Learning the rule of *hypothetical syllogism,* i.e. the statement formula

$$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r). \quad (5)$$

in primary education can be carried out with examples such as the following:

1) If the sum of the digits of the number n is divisible by 9, then n is divisible by 9.

2) If the number n is divisible by 9, then n is divisible by 3.

Conclusion. If the sum of the digits of the number n is divisible by 9, then n is divisible by 3.

Example. Learning the *law of contrapositive*, i.e. the statement formula

$$(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p) \tag{6}$$

in primary education can be carried out with examples such as the following:

If the digit of the ones of the natural number n is 0 or 5, then it is divisible by 5.

Conclusion. If the natural number n is not divisible by 5, then the digit of ones is different from 0 and 5.

Analogous concluding is an intellectual activity of making conclusions according to which if two specific subjects share certain properties or relations, then they also share other properties or relations that were not previously determined. The analogous conclusions are just probably correct, and because of this, just like the conclusions from the incomplete induction should be subject to testing, i.e. they need to be proved.

The following pattern is characteristic for the analogous concluding:

1) A has the properties $P_1, P_2, ..., P_k; Q$

2) B has the properties $P_1, P_2, ..., P_k;$

3) *Conclusion*: B has the property Q.

The use of this pattern will be presented in the following example.

Example *(analogy between a triangle and a tetrahedron).* In this example, first we are going to establish the analogy between the class of triangles and the class of tetrahedrons, and then we are going to make a conclusion based on analogy.

First, we may accept that the plane in the three-dimensional space has the role of a line in the two-dimensional space (according to the axioms for planimetrics and stereometric). In this perspective, the plane is analogous to the line. Then, we note that each triangle is limited by $3 = 2 + 1$ lines, which is the smallest number of lines that is required to form a closed and restricted figure in a plane, whereas each tetrahedron is limited by $4 = 3 + 1$ planes, which is the smallest number of planes that is required to form a closed and restricted figure in space, thus *we consider the tetrahedron to be a figure analogous to the triangle.*

Each triangle has the following properties:

i) A triangle is a convex figure;

ii) A circle can be drawn around each triangle;

iii) The symmetries of the sides of each triangle cross in a single spot, which is the centre of the drawn circle;

iv) For the area P of each triangle it is true that $P = \frac{ah}{2}$, a is the length of the base, and h is the height.

Now by analogy we get:

v) Each tetrahedron is a convex figure;

vi) We can draw a sphere around each tetrahedron;

vii) The symmetric planes of the edges of the tetrahedron (a total of 6) cross through a single spot, the centre of the drawn sphere; and

viii) For the volume V of each tetrahedron $V = \frac{BH}{3}$ is true, where B is the area of the base, and H is the height.

viii') For the volume V of each tetrahedron $V = \frac{BH}{2}$ is true ,where B is the area of the base, and H is the height;

We know that this is not true.

Analogy is a *similarity* of some sort. The essential difference between analogy and the other types of similarities lies in the intention of the person using it. If the relation in which the objects A match the objects B at a level of established notions, then these objects are treated as analogous. If we succeed to find clear connections between the objects A and B, then the analogy is clear.

Generally, when the analogy reaches a level of a logical notion, we say that the analogy is *clear* or *strong*. Otherwise, we say that it is *unclear* or *weak.* Frequently, the analogy is unclear, because the answer of the question "what is analogous to what?" is not always unequivocal. Nonetheless, the lack of clarity of the analogy does not reduce its usefulness in the instruction.

In analogy a single object is directly researched, which is then followed by making conclusions about other objects, i.e. a transfer of information from one to another object is being carried out. For this reason, the conclusions made by analogy are just probably true, and in fact analogy does not provide an answer to the question whether the conclusion is right or wrong. Clearly, the verification of the conclusion that is made by analogy should be done with other methods. The importance of analogy lies in the fact that it leads us to think about a new assumption, which may result in a new discovery.

Example (*analogy between a shark and a dolphin)*. The students in grade VII have the following knowledge about sharks:

- sharks live in water,
- they move with fins,
- they are spindleshaped,
- sharks give birth and they are fish (see fig. 2).

Fig. 2 Shark

As far as dolphins are concerned, the students know the following:

- dolphins live in water,
- they move with fins and are spindleshaped,
- dolphins also give birth (see fig. 3).

Fig. 3 Dolphin

Analogously they make a conclusion that the *dolphin is a fish*, which, of course, is wrong because the dolphin is a mammal.

The last example shows that the analogy which is not clear can be especially productive, such as the previous wrong analogy between the shark and the dolphin. On the other hand, when we used plane and spatial geometry as an example, we found analogy between the triangle and the tetrahedron that was reasonable.

3. METHODS AND ACTIVITIES FOR DEVELOPMENT OF THE SKILLS FOR PROPER CONCLUDING

Mathematics is a deductive science, and for this reason it is natural for the deductive methods to be dominant, whereas the inductive methods are dominant in other sciences. Nonetheless, the condition changes when the mathematics instruction is concerned. The inductive methods are more prominent in it. The use of the inductive methods is almost compulsory in primary education and the psychophysical abilities of the students are the basic reason for this. At this age, they are not capable to understand the deductive nature of science and the need of strong proofs to the claims.

Clearly, the importance of the deductive methods in the instruction should gradually grow, starting from grade V in primary education. Its role should become even more prominent in the upper grades of primary education, with the tendency to acquire the same role as the inductive methods in high school.

The inductive methods correspond to the basic task of inductive reasoning, which is establishment of reason-consequence relationships between subjects and phenomena. The inductive method is used in the instruction as a method for establishing logical connections between notions and claims. It is an approach for studying specific content.

The role of deduction is just as important in the instructional process as it is in specific sciences. For example, in mathematics almost all theorems, formulas and identities are formed and proved by deductive methods.

Further on, the deductive and inductive methods in the instructional process are not mutually exclusive. On the contrary, they supplement each other and it is very difficult to separate them in a "clear" form. As we have previously mentioned, the relationship between the deductive and inductive methods in the instruction primarily depends on the age of the students, their psychophysical abilities, previous knowledge, etc. Therefore, the adequate use of these methods in the instructional process in the separate instructional disciplines is very important not only for the instruction, but for the overall development of the students.

We have already mentioned that analogy has a prominent place in the instructional process, as one of the most important associative methods, and it stimulates thorough and permanent learning. For this reason, it is good for the teachers to use analogy as a method for making conclusions in each given occasion. While doing this, it is very important to stress that the conclusions made by analogy are probably correct, however, they must be subject to direct verification (proof).

Analogy had a special role in almost all important scientific discoveries. Its role is just as important in almost all school subjects, especially if combined with other scientific and instructional methods. However, it is very dangerous if used incorrectly. In many cases, the students use the conclusions acquired through analogy as if they were certainly correct, which often leads to catastrophic mistakes. Further on, we are going to present several examples from mathematics in which mistakes have been made because of the inadequate use of analogy.

Example. *i*) $\frac{ac}{bc} = \frac{a}{b}$ is correct, and by analogy the students usually write $\frac{a+c}{b+c} = \frac{a}{b}$, which, of course, is not correct.

ii) $\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$ is correct, and by analogy they usually write $\frac{a}{c} + \frac{b}{d} = \frac{a+b}{c+d}$ (?!?).

iii) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is correct, and by analogy they write $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ (?!?).

iv) $\log ab = \log a + \log b$ is correct, and by analogy they write $\log(a+b) = \log a + \log b$

(?!?).

v) $\sqrt{a^2b^2} = |ab|$ is correct, and by analogy they write $\sqrt{a^2 + b^2} = |a+b|$ (?!?).

Bearing these examples in mind, we can freely say that for the proper development of all students it is important that all teachers make constant efforts against the wrong use of analogy as a method for making conclusions. Practice shows that there are three ways to fight against the wrong use of analogy as a method for making conclusions:

a) Using counter-examples that prove that the claims acquired by analogy are not always correct.

b) By asking for an explanation of a claim, which is used as a basis for a conclusion.

c) By providing a renowned true claim, which makes it obvious that the conclusion made by analogy is wrong.

Although the students make a lot of mistakes due to the conclusions made by analogy, its importance in the instruction is enormous, especially for the development of the qualities of thinking, particularly the creative thinking of the students. The teachers who properly and skillfully use the methods of discovery through analogy, by asking adequate and proper questions, can achieve much better results in the instruction than the teachers who do not pay sufficient attention to this method. It is exceptionally important that the teachers explain the weak analogies and that they pay special attention to the ways for avoiding the wrong use of the method of making conclusions by analogy.

4. CONCLUSION

The conclusion making methods are very important for the overall development of the students, especially in the training for independent learning. For this reason, in the instructional process, the teachers should pay special attention to the learning of the conclusion making methods. Bearing in mind what we have previously elaborated, it is very important when preparing syllabi, one of the objectives of learning to be the learning of conclusion making methods (depending on the age of the students). This objective, of course, can be accomplished if the syllabus and curriculum in science are structured in such a way so as to meet the following conditions:

- modern content,
- scientific instructional content,
- structured instructional content, which means that they need to be adjusted to the age of the students and their psychophysical abilities,

- connection to the daily activities of the students and their technical orientation,
- opportunities to use this kind of knowledge.

Whether and to what extent these conditions were present in the content planned by the previous syllabi and curricula in science should be a subject of a special analysis. Nonetheless, we can freely say that the new syllabi for this group of subjects do not meet these conditions. This means that the new syllabi do not provide adequate learning of the methods for making conclusions and that they can also be an obstacle in the learning of the scientific methods and the development of the quality of thinking.

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